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The Expansion Steam Engine and the Hyperbolic Logarithm from Farey to Dixon

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In an earlier paper[1] on designing steam engines using the Soho slide rule, I briefly discussed the calculation of an expansion steam engine. In the following this topic shall be taken up again with the aim to highlight its computational treatment, especially the use of the natural logarithm. The two textbooks by Farey[2] (1827) and by Dixon[3] (1875), serve as boundaries on the time period being considered.

These books appeared after the first and before the last quarter of the 19th century, with an interval of almost 50 years. Both authors deal in detail with the calculation of steam engines and explain the use of a slide rule for this purpose.

A short explanation of the function of an expansion steam engine should make clear what role the hyperbolic or natural logarithm plays.

With several inventions and improvements, the engineer James Watt (1736 - 1819) significantly influenced the development of steam engines. Towards the end of the 18th century, he described, among other things, the expansion steam engine in patent GB17821321 from 1782.[4] In the expansion steam engine, the piston is not pressurized with the full steam pressure over the entire working stroke, but only initially over a predetermined distance. Then the inlet valve closes and the steam expands further with steadily decreasing pressure. Figure 1[5] shows the pressure curve in a working stroke assuming that the inlet valve closes after the first quarter of the piston stroke.

In the first quarter of the stroke from position 0 to position 1, the piston is pressurized with the full and constant steam pressure, normalized to p1 = 1 in the diagram.

In this section the work of the piston is calculated with pressure p1 * piston area F * travel of the piston x = 1 or pressure p1 * volume from 0 to 1. This work corresponds to the rectangular area of the diagram within positions 0 and 1.

After closing the inlet valve, the volume of steam increases as the pressure decreases. With the assumption that the steam behaves like an ideal gas according to the law of Boyle and Mariotte, pressure p * volume V = a constant, and because the area of the piston does not change, one can also write p ~ $^1/_x$ with x being the travel of the piston. In position 1 the pressure is still p1, then in position 2 it decreases to ½ for double volume, in position 3 for triple volume to ⅓, and in position 4 finally to ¼.

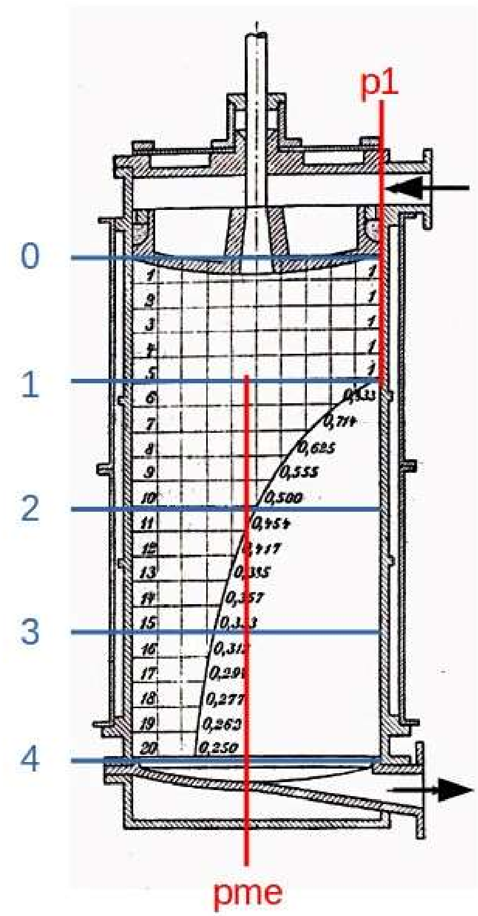


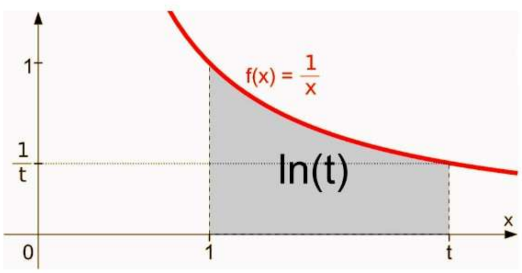
FIGURE 1. The pressure curve in the cylinder during expansion

Due to the continuously decreasing pressure from position 1, the work of the piston can no longer be calculated directly with the product pressure p * piston area F * travel of the piston x. What is needed is an assumed mean pressure, pme, during the expansion, which gives the desired result of the calculation. In other words, the area of the assumed rectangle from position 1 to 4 with the height pme must be equal to the crosshatched area under the pressure curve shown in Figure 1.

A solution results from the fact that the pressure gradient p ~ 1/x represents a hyperbola, here in a rectangular coordinate system. Then the area under the curve from x = 1 to x = t corresponds to the natural logarithm ln(t). See Figure 2.[6] With ln(4) = 1.386 one obtains the mean pressure pme = ln(4)/3 = 0.462 and the work of the piston over the whole stroke is (1*p1 + 3*pme)* piston area F = 2.386*F.

Often the piston area F = 1 is also set, because then it is easier to compare pressures and work under different conditions.

If the piston is subjected to full pressure for a fraction other than a quarter of its stroke, other numerical values will result.



$$\int_1^t \frac{1}{x}\,dx = \ln(t)$$

FIGURE 2. The hyperbola f(x) = 1/x

It should be noted that Watt does not follow this approach. He determined the mean pressure by summing up 15 ordinates as stripes of equal widths and with their respective pressures below the hyperbola.

On the basis of the calculated results, Watt points out the advantage of the expansion steam engine. If the full steam pressure is applied to the piston only over a quarter of its stroke and not over the entire length of its stroke, the coal consumption for steam generation is reduced to a quarter, but the work of the piston only to slightly more than half.

With smaller ratios of full pressure to expansion the advantage becomes even greater, but then other technical problems arise.

One restriction must be taken into account. The calculation does not exactly give the average pressure, because steam is not an ideal gas, because the temperature of the cylinder does not remain constant, and above all because opening and closing of the valves influence the pressure curve and the remaining steam in the pipe from the valve to the cylinder also plays a role. For this purpose, further calculations or correction factors based on experience with previously built machines must be applied.

To calculate an expansion steam engine, the engineer and patent attorney John Farey Jr. (1791 - 1851), used the natural logarithm. In the past, the logarithm to the base e was also called the hyperbolic logarithm because it quantifies areas under the hyperbola, as shown above.

Farey takes text passages from the Ecyclopedia Britannica into his work. Towards the end of the 18th century, not all engineers had a table of the natural logarithm:

> "As few professional engineers are possessed of a table of hyperbolic logarithms, while tables of common logarithms are, or should be, in the hands of every person who is much engaged in mechanical calculations, the following method may be practised.[7]"

The "following method" referred to above is the conversion of log(x) to ln(x) by using the multplier 2.30258 thus: ln(x) = 2.30258...*log(x). It is used in the next calculation example.

The force on the piston at maximum pressure is 6333 lbs. The stroke is 6 ft long, and after 1.5 ft the steam supply is shut off. This results in 6 /1.5 = 4; log(4) = 0.602; ln(4) = log(4) * 2.3026 = 1.386; 1 + 1.386 = 2.386; 6333 lbs * 2.386 = 15110 lbs "*Accumulated* Pressure"[8]. The result does not have a unit for work commonly used in technical fields, because the unit of length corresponds to the piston stroke without expansion.

As a working aid, Farey provides a table[9] of the so-called hyperbolic logarithms from 1 to 10 in increments of 0.05, 10 to 100 in increments of 5, plus 1000 and 10,000.

Farey explains a variety of different calculations that may occur during the design of a steam engine. He uses a slide rule of the Soho[1] type and the version he has improved. A scale of the natural logarithms on the slide rule is not mentioned anywhere in his work.

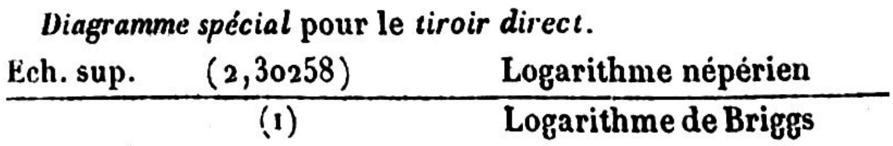


FIGURE 3. Conversion of Logarithms by Benoît

In a mathematical or technical textbook that makes use of logarithms, their definition, or at least an explanation of them, should not be missing. Farey shows that the logarithm of a number is the exponent in a power of 10 that produces exactly that number. At the same time, he uses the historical definition,

> "LOGARITHMS are a series of artificial numbers, adapted in a particular manner to a series of real numbers,..."[10]

This view had already been used by John Napier (1550 -1617), who calculated the first usable logarithm table and published it in 1614. He used it as an explanation besides the term logarithm.[11]

In the course of the 19th century, the natural logarithm is either taken from such tables or calculated from the common logarithm. The engineer Martin Benoît shows in his explanations of the slide rule from the middle of the 19th century[12] how to proceed in this case. On p. 436 he gives a simple sketch on how to set the slide rule[13], as was common at that time. See Figure 3.

Because the accuracy of the slide rule is limited to a few digits, Benoît refers to the tables by Callet.[14] Among other tables, they contain the 1 to 100 times conversion factors from base ten logarithms to natural and vice versa, given to 23 decimal places.

A scale on the slide-rule divided according natural logarithms is also not mentioned by Benoît.

In the time between Farey and Dixon little changes in this regard. It is noteworth that during this interval, in 1859, Amédée Mannheim, an officer in the French Army, and a mathematician, introduced a new slide rule design which uses the scales A,B and C,D, but in a different arrangement compared to James Watt's Soho slide rule, which was the most common type of slide rule in use at that time. The Mannheim slide rule may not have been the first slide rule to have a cursor, but it was the first to require it. Some versions of the Mannheim slide rule also have trigonometric scales, and a scale of logarithms to base 10. The natural logarithm plays almost no role.

The subject of calculating an expansion steam engine with the aid of a slide rule adapted for technical purposes is taken up again in a textbook[3] in 1875. The engineer Thomas Dixon, already mentioned above, focuses in his treatise on the novel design of his slide rule.[15] He describes the design and arrangement of the scales on it and explains its use with examples from practical mathematics, technical mechanics, and also in the design of steam engines.

First, Dixon discusses the special nature of logarithms. Like Farey before him, he draws on the historical comparison of an arithmetical sequence with geometric sequences and derives from this,

> "And logarithms being artificial numbers so contrived that the sum of the Logs. of any two numbers = the Logarithm of the Product of those numbers..."[16]

He continues with the conversion of divisions to subtractions and so on. It is astonishing how long historical interpretations survive even in mathematics. Only later does he addresse the logarithms of Briggs with their assignments: log(1)=0; log(10)=1; log(100)=2 and so on.

After this introduction he starts talking about his slide rule. The dimensions of the slide rule are 19 1/2" x 2 1/4" x 5/8" (49.5 x 5.7 x 1.95 cm).[17] In the preface Dixon mentions the company Aston and Mander in England as the manufacturer. See Figures 4 and 5.

The front face of the slide rule is labeled "No. 1", and has the following scales (from top to bottom):

CUBE ROOT 1...10 $= \sqrt[3]{N}$
(s) **N** 1...10...100...1000
A like N
(s) **B** and **C** like N
SQ.RT D 1...10...31.62 $= \sqrt{N}$

The scales marked with (s) in the above list are on 2 slides which are only visible and effective on the front side.

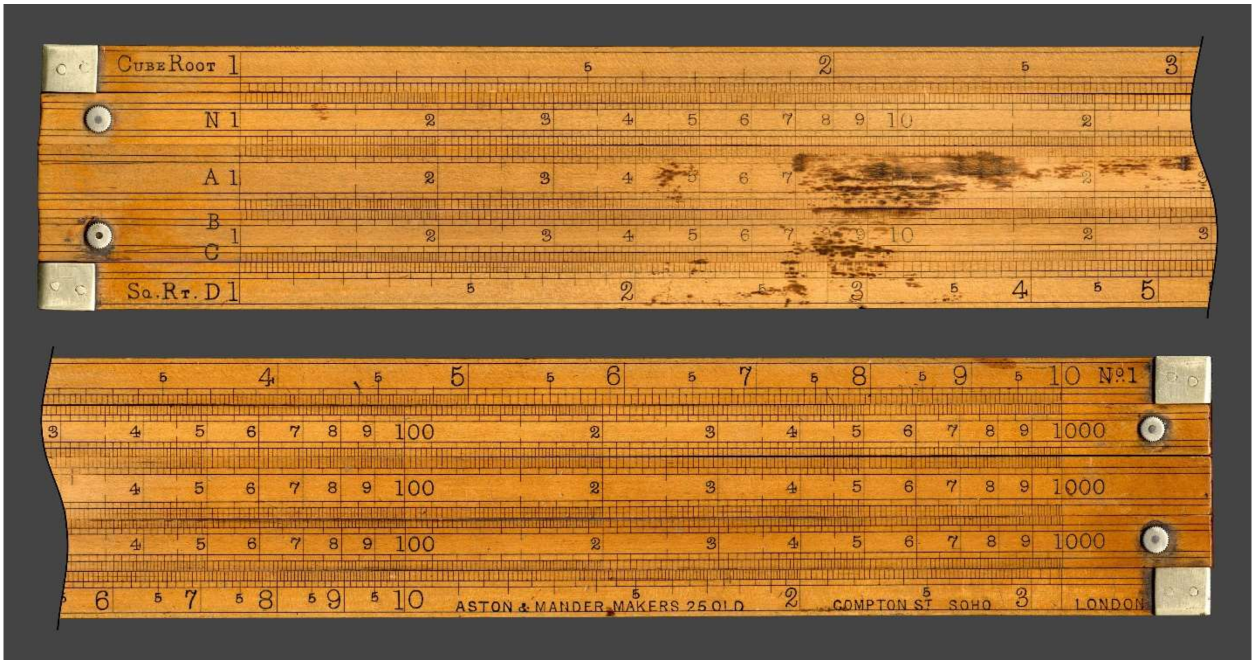


Figure 4. The Front of the Dixon Slide Rule

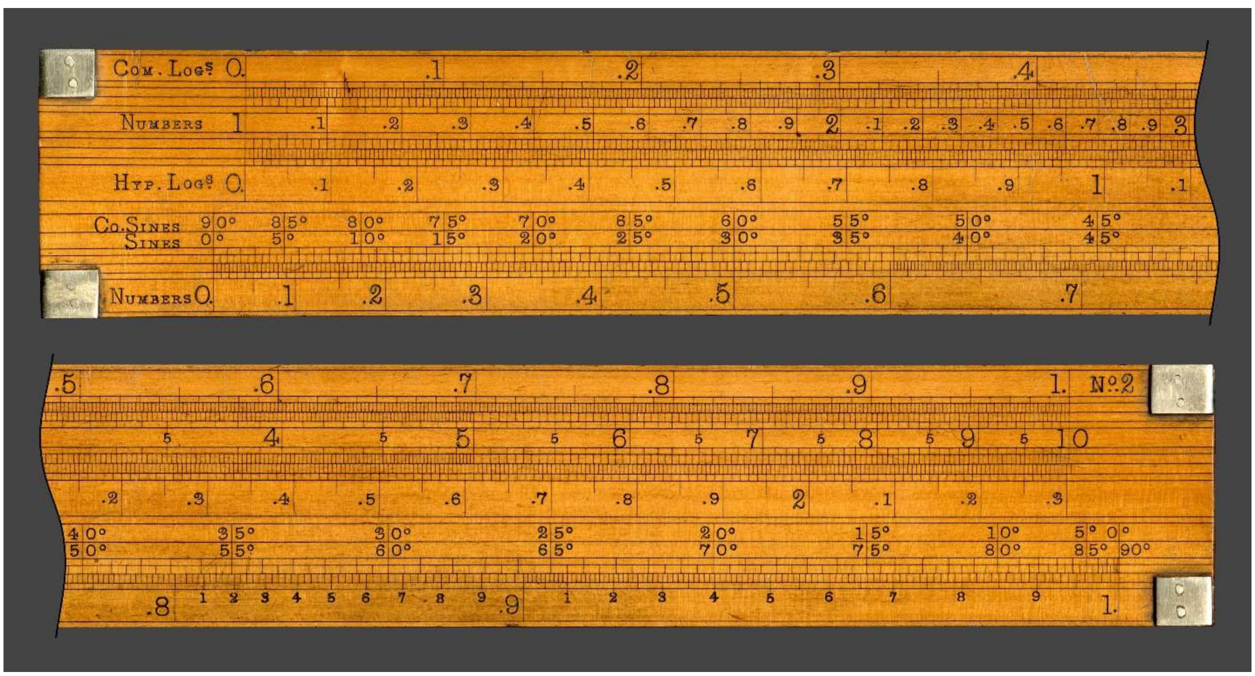


Figure 5. The Back of the Dixon Slide Rule

The back, marked "No. 2", carries from top to bottom the scales:

COM.LOGS 0...1,0 = log(NUMBERS)
NUMBERS 1...10
HYP.LOGS 0...2,3 = ln(NUMBERS)
COSINES 90°...0° = cos(), related to the scale NUMBERS below
SINES 0°...90° = sin(), same.
NUMBERS 0...1

The lower edge is labeled "No. 3" and has rulers divided in ¾ inch, 1 inch, and 1.5 inch increments. The upper edge is labeled "No. 4" and has rulers divided in ¼ inch and ½ inch increments.

With the additional scales and their arrangement, Dixon's scale layout differs significantly from the other slide rules offered at that time. This is especially true for the scales of natural and hyperbolic logarithms. Contemporary descriptions explicitly emphasize the new composition. For example, the collection catalogue of the South Kensington Museum from 1876 writes about the slide rule

> "Slide Rule, of boxwood, arranged by Mr. Dixon, Lowmoor Ironworks. Aston & Mander. In addition to the lines of the ordinary slide rule this instrument contains: Lines of common and hyperbolic logs and numbers. Lines of sines, cosines, and numbers. Lines of cubes and roots, direct."[18]

Dixon compares the design of his instrument with the existing ones and highlights the extended application possibilities:

> "...and as the arrangement now proposed and hereafter described has its lines A,B,C,D similarly marked to the Soho and Routledge, so the operations (concerning those lines only) will be alike for all three; while the lines on the proposed instrument, extra to A,B,C,D, have a special application to purposes of calculations hereafter explained, for which the other two instruments are not adapted."[19]

Some years later, a technology magazine evaluates Dixon's slide rule in an exhibition:

> "Dixon shows his 'triple radius double slide rule' with which very complex operations may be readily performed."[20]

When looking through Dixon's sample calculations, three groups of tasks stand out for which his slide rule should be suitable. These are:
1. the treatment of powers with fractional exponents, as they occur in mechanics,
2. calculations composed of multiplications, divisions and roots with several numerical values,
3. calculations in connection with the hyperbolic logarithm without intermediate calculations and without the use of a logarithm table. This is especially true for the design of expansion steam engines. For this he gives five example calculations.[21]

The first example is as follows:
The maximum pressure is 30 lbs. per sq.inch and the stroke is 60 inches. After 20 inches, the steam supply is stopped. What is the average pressure over the entire stroke?
1. 60/20, *"Or, by Slide Rule, 60 on A - 20 on N or B gives 3 on A for the number of times the steam is expanded."* At some points in the text, Dixon specifies the slide rule setting.
2. On the back of the slide rule, the value 1.098 on the scale Hyp.Log is read off for 3 on the scale Num.
3. The normalized work over the entire stroke is 1+1.098 = 2.098, thus the average pressure over the entire stroke is 2.098/3=0.7 lbs per sq.in.
4. The mean pressure over the entire stroke is 0.7*30=21 lbs. per sq.in.

The scale Hyp.Log is used like a graphical logarithm table on the slide rule, more is not possible because of its placement on the back of the slide rule, which has no slides.

As far as is known, Dixon was the first to place a scale of natural logarithms on a slide rule. This is why his work does not contain a tabular list of natural logarithms, but instead a separate chapter on hyperbolic logarithms as well as numerous examples of reading this scale.

The reason for adding the natural logarithms scale is clearly its use for the calculation of expansion steam engines.

Although no longer necessary with this new scale, Dixon also demonstrates the usual calculation of the natural logarithm:

"*Common Log. on A + 2.3 on N or B = Hyp. Log. on A.*"[22]

Besides the Soho slide rule as a whole, the scale of the natural logarithms is another example of how a slide rule can be adapted to the requirements of mechanical engineering.

In the post-Dixon era, a scale of the natural logarithms is very rarely applied to slide rules. With the beginning of the 20th century it appears in a modified version as log-log or LL scales. But that is another story.

Notes (all hyperlinks tested and working as of 21 Jan 2021)

1. Weiss, Stephan, *The Design of a Steam Engine by Means of the Soho Slide Rule*, Journal of the Oughtred Society 28:2, Fall 2019.

2. Farey, John, *A Treatise on the Steam Engine, Historical, Practical, and Descriptive*, 1827, Longman, Rees, Orme, Brown, and Green. London, https://books.google.com/books/about/A_Treatise_on_the_Steam_Engine.html?id=bfvNAAAAMAAJ
3. Dixon, Thomas, 1875, *Treatise on the Arrangement, Application, and Use of Slide Rules*, Bradford. 2ed. with supplement 1881, https://www.google.com/books/edition/Treatise_on_the_Arrangement_Application_/1jUDAAAAQAAJ?hl=en&kptab=editions&gbpv=1
4. Original Title A.D. 1782, No 1321. Specification of James Watt. – Steam Engines.
5. The picture is taken from Matschoss, Conrad, *Geschichte der Dampfmaschine*. Berlin, 1901, p. 73 and amended by the author. An identical picture, with markings, is used in the above mentioned patent for James Watt.
6. The sketch on top is taken from Wikipedia, search term *Natural Logarithm* (last visit June 6th 2020).
7. Farey, 1827, p. 343. Taken from Encyclopedia Britannica, 3rd ed., Vol. 17, (1797), search term Steam Engine
8. Farey, 1827, p. 343.
9. Farey, 1827, p. 345. With the same step interval as Bourne, John, 1851 and later, *A Treatise on the Steam Engine*. London.
10. Farey, 1827, p. 533.
11. Napier, John, *Mirifici logarithmorum canonis constructio*, 1619, Edinburgh, Positio Prima. Translated into English and annotated by Ian Bruce. http://www.17centurymaths.com/contents/napiercontents.html
12. Benoît, P(hilippe) M(artin) N(arcisse), *La Règle à Calcul Expliquée*, 1853, Paris.
13. Weiss, Stephan, 2017, *The Methodology of Teaching a Logarithmic Slide Rule in Historical Sequence* , Journal of the Oughtred Society 26:2, Fall 2017.
14. Callet, François, *Tables Portatives de Logarithmes*, 1795 and later, Paris, https://www.google.com/books/edition/Tables_portatives_de_logarithmes_contena/tY9lAAAAcAAJ?hl=en&gbpv=0
15. Wyman, Thomas, 1996, *The Thomas Dixon Engineer's Slide Rule*, Journal of the Oughtred Society, 5:2, S. 68. Pictures in International Slide Rule Museum: Aston And Mander Makers - Dixon Style Slide Rule. URL https://www.sliderulemuseum.com/Rarities.htm (last visit June 6th 2020). Pictures in Science Museum Group: T.Dixon's slide rule. URL . https://collection.sciencemuseumgroup.org.uk/objects/co60507/t-dixons-slide-rule-boxwood-19-1-2-x-2-1-4-x-slide-rule-dixon. (last visit May 7th 2020)
16. Dixon, 1875, p. 8.
17. Information about the object from Science Museum Group.
18. Catalogue of the Special Loan Collection of Scientific Apparatus at the South Kensington Museum 1876, 3rd ed., London 1877, Section 1. – Arithmetic, https://www.google.com/books/edition/Catalogue_of_the_Special_Loan_Collection/Vu8NyAEACAAJ?hl=en&gbpv=1
19. Dixon, 1875, p. 13.
20. Van Nostrand's Engineering Magazine , Vol. XXXIII, July-Dec 1885, p. 517 upper left.
21. Dixon, 1875, p. 135.
22. Dixon, 1875, p. 41.