Stephan Weiss

Symmetrical Multiplication -<br>Johann Georg Gottfried Seidel and His Multiplication Device

## Introduction

This article on the history of instrumental arithmetic presents selected methods of written multiplication with figures, which are now no longer used. Spoilt by electronic calculators for all purposes and at every opportunity, it is now hard to imagine that variants of written multiplication were worked out in previous centuries with the aim of simplification. Mechanical aids to calculation were derived from some of these.
The path to one of these aids to multiplication is traced to its presumed inventor. Later devices that apply variations of this method are also presented.

## The Multiplication of Multi-Digit Numbers

Two procedures for carrying out the written multiplication of two multi-digit numbers can be distinguished. The first consists of multiplying of one multidigit factor by all the figures of the other one after another and then adding up the intermediate results. In the second procedure, the two multi-digit numbers are directly multiplied and the result is built up during the calculation. The process of breaking the factors down into totals before the actual multiplication is not taken into account here.

We know the first way from our school days. Figure 1 shows a method for this process, as taught by Peter Apian (1495-1552) in his textbook on business arithmetic [2].


Fig. 1: multiplication procedure with multidigit times single digit in Apian

In the task shown, 987 is multiplied by 567 . The products $987 \times 7,987 \times 6$ and $987 \times 5$ are written one underneath the other, in the right places. The second product factor 567 is placed under the first and states where the units of the three intermediate results have to be and thus helps one to position them correctly.

The majority of historical multiplication aids are oriented towards the multiplication of a multi-digit product factor with a single-digit one. As the second product factor only has one digit, the structure of the multiplication procedure is simple and the subproducts can be presented without much effort. Differences exist in the presentation of the subproducts and in the carrying-over of tens from subproduct to subproduct. The best known arithmetical aids of this kind are the calculating rods of John Napier 1617 [27] and that of Genaille and Lucas 1885 [26]. Because there is already extensive specialist literature available on these, they will not be described once again here.

The second procedure is characterised by the fact that all subproducts are directly calculated and added in only one calculation operation. It comprises several different methods. The result is here built up place by place, usually starting with the units. For clarification, Fig. 2 shows which subproducts each determine a place in the result, for two three-digit factors ${ }^{1} F_{1}$ and $F_{2}$ with their hundreds $h$, tens z and units e

$$
\begin{aligned}
& \mathrm{F}_{1}=100 \times \mathrm{h}_{1}+10 \times \mathrm{z}_{1}+\mathrm{e}_{1} \\
& \mathrm{~F}_{2}=100 \times \mathrm{h}_{2}+10 \times \mathrm{z}_{2}+\mathrm{e}_{2}
\end{aligned}
$$

[^0]\[

$$
\begin{aligned}
& \left(h_{1} * 100+z_{1} * 10+e_{1}\right) * \\
& \left(h_{2} \star 100+z_{2} * 10+e_{2}\right)= \\
& =10^{4} \star h_{1} h_{2} \\
& +10^{3} \star\left(h_{1} z_{2}+h_{2} z_{1}\right) \\
& +10^{2} \star\left(h_{1} e_{2}+z_{1} z_{2}+e_{1} h_{2}\right) \\
& +10^{1} \star\left(z_{1} e_{2}+e_{1} z_{2}\right) \\
& +10^{0} \star e_{1} e_{2}
\end{aligned}
$$
\]

Fig. 2: Subproducts in the multiplication of two three-digit figures

The listing of the product in order of powers of ten shows that the units digit in the result is determined by the product of the units digits in the factors. The tens digit of the result is determined by the sum of the subproducts units digit of one factor by tens digit of the other plus tens digit of one factor by units digit of the other factor plus a possible carryover of tens from before and so on.

For better clarity, the presentation above is given showing only two three-digit factors. It can be easily generalised for any number of places (see also [15], p.8). The following applies

$$
\begin{aligned}
& \left(a_{0}+10 a_{1}+10^{2} a_{2}+10^{3} a_{3}+\ldots\right) \times\left(b_{0}+10 b_{1}+10^{2} b_{2}+10^{3} b_{3}+\ldots\right)= \\
& \left(c_{0}+10 c_{1}+10^{2} c_{2}+10^{3} c_{3}+\ldots\right) \text { with } \\
& c_{n}=b_{n} a_{0}+b_{n-1} a_{1}+b_{n-2} a_{2} \ldots+b_{0} a_{n}
\end{aligned}
$$

Of course, the carrying out the first procedure several times with several digits multiplied by a single digit also leads to this presentation, only the algorithms of the multiplication differ.

The direct multiplication of two multi-digit factors is already taught in Indian mathematics and travels from there to Europe, via Arabian works. The Italian mathematician Luca Pacioli (1445-1517) summarises the mathematical knowledge of his times and presents all methods of multiplication [17, 21, 23]. Two of far-reaching importance are presented here.

One variant of multi-digit by multi-digit multiplication is cross-multiplication, which is described in Appendix 1. It also occurs without the drawing of the crosses and is rightly described by some later authors as difficult or prone to error. No multiplication aid was created from this.

The problem in the implementation of the procedure of multi-digit by multidigit multiplication in a calculating instrument generally lies in the fact that, for all the places in the result, the sum of equivalent subproducts must be added
up while at the same time taking into account the carryover of tens from the place before.

Another written method of multi-digit by multi-digit multiplication is the socalled matrix method. It is described in Appendix 2. The matrix method is a very old multiplication procedure. Probably developed in India, it spread to China as well as - via the countries of Islam - to Europe [27]. From this characteristic arrangement of the subproducts, the Scottish mathematician John Napier (1550 -1617) developed two multiplication aids.

In Europe, the methods of direct multi-digit by multi-digit multiplication are taught into the 16th century, with little attention being paid to them afterward. In the 19th century, the method is taken up again.
In 1891, the mathematician Georg Cantor wrote
"Among the many different multiplication methods used by the Indians and which found their way into Italy (probably brought over by Arabs) from where they spread all over Europe, there were two in particular which, to be understood as opposites, still attract our attention today: crossmultiplication and matrix multiplication. In the former, no intermediate product is written and instead the final result is written down straight away; in the latter, everything is written that there is to write and the method does not omit to write e.g. the two-digit product of a multiplicator place in full into one of the multiplicands, instead of keeping the tens in one's head. Anyone who is of the opinion that most arithmetical mistakes are due to writing mistakes will still practice cross-multiplication today, which Mr. Giesing has tried to re-introduce..." [5] ${ }^{2}$

An extract from the textbook of Michael Hausbäck from the year 1833 [12] is given in Appendix 3. The author teaches the algorithm verbally and must here take into account the number of places in the two product factors, which only makes the rules of performance more complicated. His calculation instructions for two three-digit factors corresponds exactly line-for-line with the presentation in Fig. 2, starting with the units. In the foreword, the author describes the preceding method as his invention. This is of course incorrect. This procedure is later called "symmetrical multiplication" [10]. The reason for this nomenclature lies in the symmetrical appearance of the place values from both product factors in the determining of the figures in the result.

At the beginning of the 19th century, the description of a new kind of mechanical calculating aid for the direct multiplication of multi-digit figures was published. It is notable because it is not oriented towards the historical methods.

2 Translation from German.

## Johann G. G. Seidel and His Multiplier Device

The search for digitalised literature on the history of instrumental arithmetic on the internet led me to a largely unknown work. The title page is reproduced in Fig. 3. Both "Multiplikazion" with a "z" and "Rechnenmaschine" are antiquated German words from the first half of the 19th century which can easily be overlooked with an initial lack of knowledge.

When the title refers to a newly invented calculating machine one's interest is aroused. The author remains anonymous, he only refers to himself as the inventor. The encyclopaedia

Das gelehrte Teutschland oder Lexikon der jetzt lebenden teutschen
Schriftsteller, 5th Edition, Vol. 20, 1825.
gives further information. It names
SEIDEL (Johann Georg Gottfried) oldest son of Joh. Heinr. S. Accountant at the Dresden Address Office: born in Koitzsch near Trossin on 23 August 1773. §§. * Die Multiplikazion in ihrer vollkommenen Gestalt (Multiplication in its Perfect Form);... Dresd. 1823. 8. ${ }^{3}{ }^{4}$

The title of the publication is cited in full in the encyclopaedia. The attribution to the author appears reliable because the "Royal Saxon Privileged Address Office" in Dresden is named several times in Seidel's text, and also because one can buy Seidel's instrument there. The author of the work is therefore without doubt known. In Rogg's handbook of the mathematical literature 1830 [19], the work is also listed but without naming the author. Later book catalogues on the field of mathematics no longer mention the title. It is not listed in Poggendorff 1863 either. Seidel's work appears to have quickly faded into obscurity.

[^1]
# Die <br> Multiplifazion 

## in ifere volfommenfen Eefalt;

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## ふefdreibung

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## Erfitider.

## Dresben, 1823. <br> in bet Mrnoldifden कuchandiun. <br> 122.B.

Fig. 3: Seidel's Multiplikazion, cover page
"Multiplication in its most perfect form or Description of a newly invented, simple and reliable calculating machine for multiplication with multi-digit figures by means of which one finds the product of all figures, in the first line and without effort, without even knowing the times table, by multiplying oneself and with all digits at the same time. with necessary instructions for use, which include all practical advantages, previously largely unknown, in this kind of calculation, presented in summary by the inventor, for both school lessons and self-instruction. Dresden, 1823 in the Arnoldische bookshop. 122.B."

Seidel's invention consists of two components: firstly, the construction of the calculating device including the arrangement of the product factors and secondly the figures used, the number of which is increased by negative figures.


Fig. 4: Seidel's multiplying device

The calculating device has a very simple structure. It consists simply of a static table with a guide-rail and a movable strip with a grip, which is pushed into the guide-rail. On both parts, the digits of the product factors are attached at equal distances, either with the aid of written tablets or written on replaceable paper strips. Fig. 4 shows the arrangement of the digits for the product $123 \times 123=$ 15,129 . It is important here that one of the factors is attached in the reversed order of its digits. The red supplements for digits of units (e), tens (z) or hundreds (h) in the picture illustrate this special feature.
To use the device one pushes the strip from right to left until at least two digits above and below, opposite each other. In the example we get 5 positions. In each position one could multiply the digits opposite each other and add together their subproducts. In this way, one builds up the digits of the result as shown in Fig. 2 , shown starting from the units.

An example calculation is not given here because an idiosyncrasy of Seidel must be emphasised here. He does not determine the result digits by multiplication of
the factor digits. Instead of this, he is of the opinion that he can replace the multiplications of two digits and thus the whole multiplication table up to ten by a special process of continued and shortened additions. The description of the whole substitution system, and only of this, including all special cases and exceptions, takes up 105 pages in his work. This also includes the use of negative digits ${ }^{5}$. From time to time the latter were repeatedly suggested in order to achieve simplifications - or at least what the authors regarded as such - in multiplication and division. For example, Eduard Selling, Professor of Mathematics at the University of Würzburg, suggested negative digits with new numerals and even new numeral words for the use of his multiplication machine of 1887 and tried to establish these [25]. In suggestions of this kind, the additional effort necessitated by inexperienced reading and the conversion of the digits is overlooked. In addition, number systems and units of length and weight are among the stable aspects of cultures and can only be gradually changed over generations.
Like many an inventor, Seidel is obviously so enthusiastic about his invention that he makes it more and more elaborate and ignores constraints. His suggestion that the device and the replacement of the times-table also be introduced in schools was not implemented, much to our benefit.

From Seidel we learn that he also had plans for a division device that was to work similarly to his multiplication device. The announced description never appeared.

It is also to be pointed out that Seidel calls his device a machine. He is here using the word "calculating machine (Rechenmaschine ${ }^{6}$ )" in the very broad way it was used until into the 19th century, which was not made more precise by the condition that there also be automatic carryover of the tens in the result (not included in Seidel's device) until the beginning of the 20th century [29].

## Priority

Seidel calls his multiplication aid, combined with the special calculation method that belongs to it, his own invention. As regards the calculating method, this claim is incorrect. As far as the device itself is concerned, there is no record of a device that is similar in its functioning from the time of publication of his 1823 work. The manufacturer and marketer of instruments, Johann Conrad Gütle, does not describe anything that would correspond to Seidel's device in his

[^2]catalogues towards the end of the 18th century [28]. Johann Paul Bischoff, who endeavoured to find all calculating devices and machines for his encyclopaedia of 1804, also names nothing comparable [3]. Neither do mathematical dictionaries from this period give any indication of a similar device. To this extent, one must assume that priority for a symmetrical multiplication calculating machine lies with Seidel. According to present knowledge it cannot be determined from which source Seidel drew the idea or whether he originated it. The scope of priority itself needs to be made more precise, because there are predecessors.

In the mid-17th century, William Oughtred already describes the arrangement of the two factors one above the other, in which in one of them the sequence of the digits is reversed. No shifting takes place however. The arrangement is much rather for achieving an approximate result [16, p.9].

John Colson (1680-1760), Professor for Mathematics at the University of Cambridge, published a work on the multiplication of large numbers in 1726 [7]. In this, he uses negative digits, which he identified with over-scoring, and two pieces of paper that he moves against each other. One piece bears the first product factor, the movable multiplier with reversed sequence of the digits, and the second bears the multiplicand, the second product factor. ${ }^{7}$ Then he shifts one of the two pieces of paper in relation with the other and adds the subproducts of all digits that are opposite each other. Fig. 5 shows the start of multiplication with the factors
$8605729398715 \times 3894175836438$ bzw. $1 \overline{14} 1 \overline{43} 3 \overline{1} 40 \overline{13} 15 \times 4 \overline{11} 2 \overline{242} 4 \overline{4} 44 \overline{2}$


Fig. 5: Symmetrical multiplication in Colson

[^3]The multiplication here starts with the highest places in the factors and requires notation of the intermediate results because carrying-over of tens that is to occur later on in the procedure. He also carries out the process, which starts with the units, and the other basic arithmetic operations with negative digits.
The simplification of the calculations is meant to consist in the fact that the higher value digits $9,8,7,6$ are replaced by the lower-value $1 \overline{1}, 1 \overline{2}, 1 \overline{3}, 1 \overline{4}$. With these substitutions, the small multiplication table could be restricted to products up to $5 \times 5$ but additionally with taking into account of the sign. One difficulty is that it must always convert numbers with positive digits into numbers with negative digits and vice versa.
At the end of the text, Colson mentions a calculating device with the name abacus or counting table, which he has developed and wants to present soon. Unfortunately, a description has never appeared.

The use of two sheets of paper that bear product factors and the positions of which, in relation to each other, can be changed as wished represents a mechanisation of numerical calculation, albeit at a simple level. It offers a constantly changing view of configurations that cannot be achieved by notation alone. The advantage of Seidel's invention is that it applies this principle as a device that exists independently and remains re-usable.

In the 19th century, the use of two pieces of paper continues to be known, as the following sources prove.
In the German translation of the work of the French mathematician and physician Jean Bapiste Fourier (1768-1830) Analyse des équations déterminées from the year 1831, the editor and mathematician Alfred Loewy makes an interesting comment [9, p. 262]. He writes
"61) On p. 183. The multiplication taught here by Fourier is called symmetrical... As far as we know, Fourier was the first European mathematician in whose works on symmetrical multiplication one finds the practical instruction to write the two numbers on two separate sheets and with the numbers in reverse order...."

Fourier's explanations of the procedure are not unambiguously clear. If, however, he writes the two product factors on two sheets of paper, that can only be for the purpose of shifting the two against each other. One does not need two sheets for static cross-multiplication. Lüroth makes a comment upon this method of multiplication to the effect that the two factors, one of which has a reversed arrangement of its digits, are actually shifted against each other by means of strips of paper [15]. Fourier, however, does not add up the subproducts as a whole, he only adds up their units and adds the tens of precisely these subproducts onto the next totalling. Ultimately, it is unimportant how the subproducts are added. What remains decisive about this procedure is the manual shifting of the factors against each other.

In 1840, the French mathematician Augustin Louis Cauchy writes that one can use a ruler or a strip of paper for this kind of multiplication [6, p. 437].

In 1885, Cantor, in a review of the work of Giesing of 1884, whose calculating device is looked at further below, also refers to symmetrical multiplication by means of a second piece of paper that is moved [11]. He adds that he has got to know this procedure in the winter semester 1849/50 at the University of Göttingen with Professor M. Stern ${ }^{8}$. Cantor does not mention a multiplication device of the same construction before Giesing.

In an article in 1863, William Henry Oakes suggests a method of multiplication that, as he mentions, could be carried out in one's head [30]. This is symmetrical multiplication, he merely does not call it this. The calculation example given is presented in detail and reproduced in Appendix 5.

From this, it can be concluded that symmetrical multiplication by means of factors that could be shifted against each other, with one of these factors having its figures in reverse sequence, must have been known and common in specialist circles from 1730 at the latest to at least 1900.

From the end of the 19th and beginning of the 20th century, several multiplication devices were designed and offered that were based upon the principle of symmetrical multiplication with factors that were shifted against each other.

## Later Multiplication Devices

These early devices include Poppe's Arithmograph, described in Dinglers Polytechnisches Journal of 1877 [31]. Fig 6.1 is taken from this. Multiplication rods for the digits 0 to 9 are kept in box C. From top to bottom, these rods bear the products of the headcount with 9 to 1 . The digits of these subproducts are placed at the edge of the rods, so that they can be clearly visually connected to a digit on the adjacent rod. The arrangement of the digits - units on the left and tens on the right as here or reversed - depends on the direction in which the second product factor is offset over the first.
For the calculating example $907 \times 83=75,281$ from Graf's article, one first places on the factor 83 , with the digits in reverse order i.e. as 38 , onto support A. ${ }^{9}$

[^4]Poppe's Arithmograph. (22) 1/2 nat. Gn



Fig. 6.1: Poppe's Arithmograph, top: the device, bottom: positions of the calculating example

On the slide B, the second product factor is set by vertical shifting of strips with display openings. These show openings are then at the height of the subproducts and give an unobstructed view of two adjacent figures on two rods. Then, one moves the slider from left to right over the rods that have been placed on, reads off the digits displayed in each position and adds them up. The totals each produce a digit in the desired result. On the arithmograph, the calculating example mentioned gives the displays shown below in Fig. 6.1. The author of the article explains this in footnote 2

The product $907 \times 83$ produces the following subproducts:

| 7 | units | $\times$ | 3 | units | 00021 |
| :--- | :--- | :---: | :---: | :---: | :--- |
|  | tens | $\times$ | 3 | $"$ | 0000 |
| 9 | hundreds | $\times$ | 3 | $"$ | 027 |
| 7 | units | $\times$ | 8 | tens | 0056 |
|  | tens | $\times$ | 8 | $"$ | 000 |
| 9 | hundreds | $\times$ | 8 | $"$ | 72 |

Total 75281

The individual pulls are thus equivalent to
Pull I = total of all units
„ II = , tens
" $\mathrm{III}=\quad$, hundreds
" IV $=$ " thousands
, $\mathrm{V}=$, ten-thousands
Only three years later, in 1880, John Bridge publishes the description of multiplication apparatus that functions almost identically to the arithmograph [32]. Fig. 6.2 shows his diagrams of this. In the introduction he refers to Napier's calculating rods and wants to show that these have even more possible applications than have been made use of thus far. ${ }^{10}$
The article of Bridge is detailed and worth reading because it also deals with the use of such an apparatus for division and square roots.

[^5]Phil. Mag. S. 5.Vol.9.PI.VI.


Fig.6.2: Bridge's multiplication apparatus

In 1883, Karl Julius Giesing is granted a German patent for a calculating device [8], appendix 4 cites the patent claim. His Neuer Unterricht in der Schnell-rechen-Kunst (New Lessons in the Art of Swift Arithmetic) including symmetrical multiplication, which also contains a description of his invention in its second part, appears the same year [11]. In 1886, Dinglers Polytechnisches Journal gives a further description of the device [1].

Giesing (1848-1907) worked as a teacher for languages, mathematics and physics and later as a headmaster at Realschule (middle) schools [14] and was thus highly familiar with the procedure of numerical calculation.

The core of his design, as with the two above-named inventors, is that he mechanises symmetrical multiplication. The two digits to be multiplied are written on a fixed element and on a movable element and are shifted against each other. Figure 6 shows the calculating example $352 \times 436$. The factor 352 is on the base plate A, the factor 436 is shown in reverse order as 634 on the movable slider C.

Fig. 1.


Fig. 6.3: Giesing's calculation apparatus

One moves the slider over the base plate from the right. In each position in which two digits are opposite each other, their subproducts are calculated and added and - with a possible carryover from before - produce a digit in the result.

The calculation in Fig. 6 proceeds in the following five steps:

|  | Position | Calculation | Digit <br> Result | Carry over |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $\begin{gathered} 352 \\ \leftarrow \\ \leftarrow \\ \\ 634 \end{gathered}$ | $2 \times 6=12$ | 2 | 1 |
| 2 | $\begin{array}{llll} 3 & 5 & 2 \\ & 6 & 3 & 4 \end{array}$ | $1+6 \times 5+3 \times 2=37$ | 7 | 3 |
| 3 | $\begin{array}{lll} 3 & 5 & 2 \\ 6 & 3 & 4 \end{array}$ | $3+6 \times 3+3 \times 5+4 \times 2=44$ | 4 | 4 |
| 4 | $\begin{array}{rlll}  & 35 & 2 \\ 6 & 3 & 4 \end{array}$ | $4+3 \times 3+4 \times 5=33$ | 3 | 3 |
| 5 | $\begin{array}{rlrl}  & & 352 \\ 6 & 3 \end{array}$ | $3+4 \times 3=15$ | 5 | 1 |
|  |  |  | 1 |  |

The result: $352 \times 436=153,472$
Intermediate results can be noted down in the columns marked with small letters, which are attached to the base plate.
Like Seidel, Giesing also resorts to negative digits:
"In order to avoid larger totals of the products to be added up, one sometimes increases the number before an 8 or 9 by a unit and puts in place of those numbers $\overline{2}$ resp. $\overline{1}$,which is supposed to indicate that the products generated by multiplication with $\overline{2}$ or $\overline{1}$ are to be subtracted." [8, p. 2 top]
For example, he replaces the factor 5,396 by $5,4 \overline{1} 6$. Numbers with negative digits are not absolutely necessary for the calculation process.

Symmetrical division is also based upon a process taught by Fourier and differs substantially from normal division, because only some digits are taken into account instead of the whole divisor [15, p. 38 ff$]^{11}$. In the patent specification, Giesing gives a detailed calculation example of division and also of the calculation of the root using his device.
I am not aware of a built version of the multiplication device.
The pocket device Multor or Multirex, produced by the company Ludwig Spitz \& Co. in Vienna, is among the few multiplication aids that have actually been made. The device was on sale at the beginning of the 20th century. The names Universalrechner (universal calculator) or Zauberapparat (magic device) also used are to be regarded as nothing other than advertising slogans intended to exaggerate the efficacy of the device.

[^6]

Fig. 7: Multor or Multirex

The device is small in construction at 13 by 8 cm and consists of a fixed lower part over which a movable part can be perpendicularly moved (Fig. 7).
A rod, attached at the very bottom in the picture, makes it easier to adjust the numbers.


Fig. 8: Multor, the first three positions for $352 \times 436$

For multiplication, one of the product factors is set on the right on the sliders in the base plate. This makes the multiples of this digit, by 1 to 9 , appear next to each other. One setsthe second factor on the movable part by opening little windows marked with the digits 1 to 9 . They reveal the rows of multiples lying
below. In Fig. 8, for clarification, the calculation $352 \times 436$ is portrayed with the factors in blue and in the first three positions from the top to the bottom.
Unlike in Giesing's patent, the digits of the factors are not multiplied. Instead of this, in each position of the upper movable part the device shows all digits of those subproducts that produce a digit of the result when added together. The user does not have to multiply, but instead only add up the digits that are shown and any carryovers from before. In this respect, this device goes a step further than that of Giesing as regards simplification.
The following demonstration shows all the steps of multiplication $352 \times 436$ with the respective subproducts and the digits of the subproducts shown in each position.

| Step $\rightarrow$ | 6 | 5 | 4 | $\mathbf{3}$ | $\mathbf{2}$ | $\mathbf{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Subprod. $\downarrow$ | $10^{5}$ | $10^{4}$ | $10^{3}$ | $10^{2}$ | $10^{1}$ | $10^{0}$ |
| $2 \times 6$ |  |  |  |  | 1 | 2 |
| $6 \times 5$ |  |  |  | 3 | 0 |  |
| $3 \times 2$ |  |  |  |  | 6 |  |
| $6 \times 3$ |  |  | 1 | 8 |  |  |
| $3 \times 5$ |  |  | 1 | 5 |  |  |
| $4 \times 2$ |  |  |  | 8 |  |  |
| $3 \times 3$ |  |  | 9 |  |  |  |
| $4 \times 5$ |  | 2 | 0 |  |  |  |
| $4 \times 3$ | 1 | 2 |  |  |  |  |
| Result $\rightarrow$ | 1 | 5 | 3 | 4 | 7 | 2 |

The calculating machine La Multi, of French origin, which functions in the same way, was on sale in the 1920s (Fig. 9). A contemporary description can be found in the journal La Nature from the year 1920 [4]. Here, the movable frame is moved horizontally. Due to its size it is intended for stationary use at the workplace.


Fig. 9: La Multi

As regards the sales success of the two latter commercial devices, it can only be surmised that it cannot have been very great, in view of the lack of repeated attention in the contemporary literature.
Neither the inventors nor the producers of the devices make any kind of reference to predecessors of this principle.

## Appendix 1: Cross-multiplication According to Pacioli

Pacioli counted cross-multiplication as the fourth method of multiplication. He calls it crocetta (small cross) or casella (small house, also drawer), in the original De $4^{\circ}$. mõ multiplicandi dicto crocetta siue casella. Fig. 10 shows the procedure for $456 \times 456$. The sequence of the partial multiplication has here been marked afterwards with blue numbers.


Fig. 10: Cross-multiplication with Pacioli

The calculation runs as follows:

|  | Place in <br> the result | Subproducts | Digit <br> Produced | Carry <br> over |
| :---: | :---: | :--- | :---: | :---: |
| 1. | $10^{0}$ | $6 \times 6=36$ | 6 | 3 |
| 2. | $10^{1}$ | $3+5 \times 6+5 \times 6=63$ | 3 | 6 |
| 3. | $10^{2}$ | $6+4 \times 6+4 \times 6+5 \times 5$ <br> $=79$ | 9 | 7 |
| 4. | $10^{3}$ | $7+4 \times 5+4 \times 5=47$ | 7 | 4 |
| 5. | $10^{4}$ | $4+4 \times 4=20$ | 20 | - |

The result: $456 \times 456=207,936$

## Appendix 2: The Matrix Method of Multiplication

The matrix method of multiplication arranges both product factors at the sides of a rectangle which is in turn divided into small squares. Fig. 11 shows the matrix method, also called gelosia or graticola, as shown in Pacioli for the calculation $987 \times 987$. The presentation in Fig. 12 in Regius [18] is more clearly laid out with distinct demarcation from the result. The places of the product factors 468 and 246 have here been marked in red afterwards for greater clarity. The squares are also divided by diagonal lines.


Fig. 11: Matrix multiplication in Pacioli


Fig. 12: Matrix multiplication in Regius


Fig. 13: Left and below, a horizontal and a vertical strip from the Promptuarium by Napier

After the writing of the product factors, the subproduct of the two outside digits is written into each square. The digits of the subproduct are separated by diagonal lines. As the next step, all the digits of a diagonal are added together, starting on the far right with the units. The unit digit of each intermediate
result forms a digit of the desired result, the tens digit is carried over to the next diagonal addition. A symmetrical arrangement with the product factors at the top and left instead of right and with diagonals that run from the top right to the bottom left also occurs.

Napier developed two multiplication aids from this two-dimensional arrangement of subproducts. They are his famous multiplying rods of the Rabdologia (rod calculation) for the multiplication of multi-digit figure by a single-digit figure and the Promptuarium multiplicationis for the multiplication of two multi-digit figures. The latter consists of horizontal and vertical strips which, placed one above the other, present the arrangement of matrix multiplication (Fig. 13) [13, 27].

## Appendix 3: A New Way of Multiplying Figures

Source: Hausbäck, Michael [12], 1833, p. $10^{12}$
"In multiplication with 3 digits in the multiplicator and in the multiplicand, the product contains 5 or 6 digits and one receives

1) One gets the units by multiplying unit by unit, the units, or if they are insufficient, by writing a zero and keeping the tens for the next place.
2) One gets the tens by multiplying the units of the multiplicator by the tens of the multiplicand and the tens of the multiplicator by the units of the multiplicand and adding the tens that remain in No. 1 to the sum of both products (in order not to forget them, one can also add the tens remaining in No. 1 to the first product straight away and then add on the other product.) One then writes the tens and saves the hundreds for the next place.
3) One gets the hundreds by multiplying the units of the multiplicator by the hundreds of the multiplicand and the tens of the multiplicator by the tens of the multiplicand, and the hundreds of the multiplicator with the units of the multiplicand and adding the hundreds remaining in No. 2 to the sum of these three products. One writes the hundreds and keeps the thousands for the next place.
4) One gets the thousands by multiplying the tens of the multiplicator by the hundreds of the multiplicand, and the hundreds of the multiplicator by the tens of the multiplicand and adding the thousands remaining in No. 3 to the sum of both products. One writes the thousands and keeps the ten thousands for the next place.
5) One gets the ten-thousands by multiplying the hundreds of the multiplicator by the hundreds of the multiplicand and adding to that the tenthousands that remain in No. 4."

12 Translation from German source.


## Appendix 4: Patent Specification DE26107

The title of the patent specification: C. Julius Giesing in Döbeln: Calculating Apparatus. Patented in the German Empire from 31 July 1883 (excerpt): ${ }^{13}$


## PATENTCLAIM

A calculating device in the form of a board with two fixed writing surfaces A and B - divided by a straight line into columns - for task and result, and a horizontal or circular strip $C$ or rings with writing surface for the multiplicator and/or divisor inserted between these. By moving this strip C in even intervals, determined by the distances of the digits, in the direction:
a) from right to left, in multiplication, the digits of the $n$-digit multiplicator written on $C$ in reverse order of digits is placed symmetrically together in the order 1-, 2-, $3-\ldots n$ - (in $[\mathrm{m}-\mathrm{n}+1]$ times repetition), ( $\mathrm{n}-1-$ ), ( $\mathrm{n}-2$ )... with the 3 -, 2-, 1-pair groups) of the m-digit multiplicand written on $A$, producing, through multiplication of the factor pairs in the columns belonging together by group and by addition of the thus produced products of each group all value digits of the total product to be written on $B$, without written assisting calculations;
b) from left to right, in division, the first digit of the $n$-digit divisor written on $C$ in reverse digit order is moved as solely acting divisor in order below each place of the dividend written on $A$, and the first following digits are placed together with total number of digits of the quotients at $1-, 2-, 3-\ldots$ n-pair groups occurring after each individual division, the product totals of which are to be deducted for the obtaining of the new dividend and a new place in the quotient of the rest of the preceding division, after this has been broken down into units of the next order down and combined with the following place of the dividend;
c) from left to right, in the calculation of the square root, after finding of the first place of the root according to the common procedure, the double of the same is moved as a divisor in order below the remaining digits of the square root, in which one is to proceed in accordance with the division method below b ), in which each new place of the root $B$ that is found through division is at the same time placed by the divisor $C$ at the left.

Appendix 5: A Method of Multiplication which may be practised Mentally

Source: Oakes [30],

## A Method of Multiplication which may be practised Mentally. By Lieut.-Col. Wm. Henry Oakes, Bengal Presidency.

1. Write down the multiplicand in the usual manner.
2. Write on a separate slip the multiplier with its figures reversed (as in contracted multiplication), being careful to space the figures so that they can be exactly placed over those of the multiplicand.
3. Thus arranged, place the multiplier so that the figure in the true unit's place may fall over the unit's place of the multiplicand; then multiply each figure of the multiplicand by that immediately over it, sum the products and write down the last figure of the sum, reserving the rest as a carriage.
4. Shift the multiplier one place to the left, multiply each figure as before and sum the products, adding thereto the carriage from the preceding operation; write down the last figure of this new sum, and reserve the rest as a carriage.
5. Proceed in this manner until the multiplier is exhausted ; the sum last obtained must be written in full, and the work will then be complete.

$$
\text { Example. }-9763 \times 8452 \text {. }
$$

Process in detail, showing the several positions of the multiplier.
2548 Multiplier reversed.
9763 Multiplicand.
6


## Figures

1 Apian [2], section Multiplicatio
2 created by the author
3 Seidel [20]
4 Seidel [20], amended by the author
5 Colson [7], p. 166
6.1 Graf [31], text and table IV
6.2 Bridge [32]
6.3 Pat. DE26107
$7,8,9$ created by the author
10 Pacioli [17], fol. 27v, completed by the author
11 Pacioli [17], fol. 28v
12 Regius [18] 1536, fol. LVI v, completed by the autor
13 John Napier (Neper), Rabdologia, 1617, p. 94f

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[^0]:    1 It is here a precondition that the numbers are in the decimal system. The connections displayed of course apply to other base numbers. Lüroth generalises the derivations for any base numbers [15].

[^1]:    3 Citation translated from German. The superlative "most perfect" is used in the original title. Because perfection refers to a state that cannot be further improved, there is no superlative for the most perfect. The author first writes simply
    "The common art of calculating figures or so-called arithmetic has not climbed to the peak of its perfection, which consists of finding the result of any calculation in the shortest way that is at all possible." Afterwards, however, he implies that his method surpasses the previous by far in its simplicity and speed. One could come to the conclusion that he has used the superlative deliberately.
    4 "Address office (Adreß-Comptoir)... This was an intermediation office between potential buyers and potential sellers - sometimes everyday goods were also marketed there." (From Stadtwiki Dresden). In addition:
    "Address offices were places for conveying information and were intended to allow access to city resources that were otherwise hidden in obscurity and were primarily for the mediation of sales, labour, real estate and capital." [22]

[^2]:    5 An example of negative digits from Seidel, they are placed in brackets: $2(2) 2=2 \times 100-2 \times 10+2=182$.
    6 Seidel gives a definition:
    "Because I regard the term "machine" as referring to any artwork that must be used in accordance with some law of mechanics or the use of which is based upon some kind of regular movement that changes the relationship of its parts to each other. This is what actually distinguishes a machine from a mere instrument or tools" [20], p.116.

[^3]:    7 The original text reads "Write down these two numbers one under the other upon a flip of Paper, with the Figures at equal distances, and then cut them asunder. Take either of the Numbers for a Multiplier, and place it over the other in an inverted position, so as its first Figure may be just over the first Figure of the Multiplicand." [7, p.165]

[^4]:    8 Moritz (also Moriz) Abraham Stern (1807-1894) taught in Göttingen until 1884.
    9 The comment in Graf that this reversal of the sequence of the figures has nothing to do with the theory of the multiplication apparatus, but instead occurs for reasons that arise due to the desired manageability, is incorrect.

[^5]:    10 Multiplication tables with products of the small multiplication table have frequently been called Napier's rods, although the arrangement of the product digits no longer has anything to do with the characteristic arrangement found in Napier.

[^6]:    11 Towards the end of the 19th century Unger does not ascribe any practical value to Fourier's rule of ordered division and only cites it for reasons "of historical interest" [24].

